

A Simple Number Theoretic Result

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Abstract. We derive an interesting congruence relation motivated by an Indian Olympiad problem. We give three different proofs of the theorem and mention a few interesting related results.

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1. AN INTERESTING OLYMPIAD PROBLEM

Theorem 1.1. *7 divides $\binom{n}{7} - \lfloor \frac{n}{7} \rfloor$, $\forall n \in \mathbb{N}$.*

The above appeared as a problem in the Regional Mathematical Olympiad, India in 2005. Later, in 2007, a similar type of problem was set in the undergraduate admission test of Chennai Mathematical Institute, a premier research institute of India where 7 was replaced by 3.

In late 2008, the first author posted the following theorem as a question in an internet forum called Mathlinks, [7].

Theorem 1.2. *If p is any prime then, p divides $\binom{n}{p} - \lfloor \frac{n}{p} \rfloor$, $\forall n \in \mathbb{N}$.*

The second author replied to the post and proved the above result using Wilson's theorem. However, his proof was not entirely correct and together the authors managed to correct the argument.

Then we wondered what would be the case if p is not a prime.

In early 2009, we showed that if p is a composite number of the form $q^x.k$, where q is a prime and $\gcd(q, k) = 1$, then the above statement is not true. Note here that x and k cannot simultaneously be equal to 1.

Thus **Theorem 1.2** became the basis of the following theorem.

2. MAIN RESULT

Theorem 2.1. *A natural number $p > 1$ is a prime if and only if $\binom{n}{p} - \lfloor \frac{n}{p} \rfloor$ is divisible by p for every non-negative n , where $\binom{n}{p}$ is the number of different ways in which we can choose p out of n elements and $\lfloor x \rfloor$ is the greatest integer not exceeding the real number x .*

We have found three different proofs of the above result, two purely number theoretic and one via a combinatorial argument.

We also state and prove a famous theorem in Number Theory called Lucas' Theorem,

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Theorem 2.2. ([3], *E. Lucas, 1878*) Let p be a prime and m and n be two integers considered in the following way,

$$\begin{aligned} m &= a_k p^k + a_{k-1} p^{k-1} + \dots + a_1 p + a_0 \\ n &= b_l p^l + b_{l-1} p^{l-1} + \dots + b_1 p + b_0, \end{aligned}$$

where all a_i and b_j are non-negative integers less than p . Then,

$$\binom{m}{n} = \binom{a_k p^k + a_{k-1} p^{k-1} + \dots + a_1 p + a_0}{b_l p^l + b_{l-1} p^{l-1} + \dots + b_1 p + b_0} \equiv \prod_{i=0}^{\max(k,l)} \binom{a_i}{b_i}.$$

There are numerous proofs of the above theorem, the first being given by Lucas himself. We give a slightly modified version of a proof given by N. J. Fine, [2].

Proof. All we need to prove is that if p is a prime then $\binom{ap+b}{cp+d} = \binom{a}{c} \binom{b}{d}$, for non negative a, b, c, d where $b, d < p$. Once we have this, then we can use induction on it to get the desired result.

We have, for every integer k , such that $0 < k < p$ we have $p \mid \binom{p}{k}$, so we can conclude that for every integer x we have,

$$(1+x)^p \equiv (1+x^p) \pmod{p}.$$

So, from the above result we can easily see that,

$$(1+x)^{ap+b} = ((1+x)^p)^a (1+x)^b \equiv (1+x^p)^a (1+x)^b \pmod{p}.$$

Comparing the polynomial coefficients of x^{cp+d} in both congruent polynomials, we get,

$$\binom{ap+b}{cp+d} = \binom{a}{c} \binom{b}{d}.$$

This completes the proof. □

Using Lucas' Theorem, we can easily prove our result, if we consider $n = ap + b$, where p is a prime and $b < p$.

We now give a combinatorial proof of **Theorem 2.1**,

Proof. We suppose $n = pq + r$, and let us have n compartments with partitions after every p compartments. In all there are $q + 1$ partitions where the last partition has r compartments. It is assumed that each compartment is labelled and distinct. Obviously, $\binom{n}{p}$ is the number of ways to select p compartments out of n compartments.

The total number of selections thus can be divided into two sets:

- Compartments of only one partition are completely selected.
- Compartments of only one partition are not selected.

It is easy to see that there are $q = \lfloor \frac{n}{p} \rfloor$ selections of the first set.

In the second set we subdivide the selections on the basis of number of selections in each partition. Let a_i compartments be selected in the i -th partition. For each subdivision, the number of selections are $\binom{r}{a_{q+1}} \prod_{i=1}^q \binom{p}{a_i}$.

Since $a_i < p$ and at least one $a_i (1 \leq i \leq q)$ will be non zero and $\binom{p}{k}$ is divisible by p for $0 < k < p$.

Hence, each of the subdivisions of the second set will be divisible by p .

This completes the proof. \square

We give now the final proof of **Theorem 2.1** using purely number theoretic tools.

Proof. First assume that p is prime. Now we consider n as $n = ap + b$ where a is a non-negative integer and b an integer $0 \leq b < p$. Obviously,

$$\lfloor \frac{n}{p} \rfloor = \lfloor \frac{ap+b}{p} \rfloor \equiv a \pmod{p}.$$

Now let us calculate $\binom{n}{p} \pmod{p}$.

$$\begin{aligned} \binom{n}{p} &= \binom{ap+b}{p} \\ &= \frac{(ap+b) \cdot (ap+b-1) \cdots (ap+1) \cdot ap \cdot (ap-1) \cdots (ap+b-p+1)}{p \cdot (p-1) \cdots 2 \cdot 1} \\ &= \frac{a \cdot (ap+b) \cdot (ap+b-1) \cdots (ap+1) \cdot (ap-1) \cdots (ap+b-p+1)}{(p-1) \cdot (p-2) \cdots 2 \cdot 1} \end{aligned}$$

We denote this number by X .

We have $X \equiv c \pmod{p}$ for some $0 \leq c < p$. Consequently taking modulo p , we have

$$c(p-1)! = X(p-1)! = a(ap+b) \cdots (ap+1)(ap-1) \cdots (ap+b-p+1)$$

All the numbers $ap+b, \dots, ap+b+1-p$ (other than ap) are relatively prime to p and obviously none differ more than p so they make a reduced residue system modulo p , meaning we have mod p ,

$$(p-1)! = (ap+b) \cdot (ap+b-1) \cdots (ap+1) \cdot (ap-1) \cdot (ap+b-p+1)$$

both sides of the equation being relatively prime to p so we can deduce $X \equiv c \equiv a \pmod{p}$. And finally $\binom{n}{p} \equiv X \equiv a \equiv \lfloor \frac{n}{p} \rfloor \pmod{p}$.

To complete the other part of the theorem we must construct a counterexample for every composite number p . If p is composite we can consider it as $q^x \cdot k$ where q is some prime factor of p , x its exponent and k the part of p that is relatively prime to q (x and k cannot be simultaneously 1 or p is prime). We can obtain a counterexample by taking $n = p + q = q^x k + q$ will make a counter example. We have:

$$\binom{p+q}{p} = \binom{p+q}{q} = \frac{(q^x k + q)(q^x k + q - 1) \cdots (q^x k + 1)}{q!}$$

Which after simplifying the fraction equals: $(q^{x-1}k + 1) \frac{(q^x k + q - 1) \cdots (q^x k + 1)}{(q-1)!}$. Similary as above we have $(q^x k + q - 1) \cdots (q^x k + 1) = (q-1)! \neq 0$ modulo q^x therefore,

$$\frac{(q^x k + q - 1) \cdots (q^x k + 1)}{(q-1)!} \equiv 1 \pmod{q^x}$$

and

$$\binom{p+q}{p} \equiv q^{x-1}k + 1 \pmod{q^x}.$$

On the other hand obviously,

$$\lfloor \frac{q^x k + q}{q^x k} \rfloor \equiv 0 \pmod{q^x}.$$

And since $q^{x-1}k + 1$ can never be equal to 0 modulo q^x we see that

$$\binom{p+q}{p} \not\equiv \lfloor \frac{p+q}{p} \rfloor \pmod{q^x}$$

consequently also incongruent modulo $p = q^x k$.

□

Remark 2.3. Here we would like to comment that by taking q as the minimal prime factor of p and using the same method as above we can simplify the proof even more. We can then compare $\lfloor \frac{p+q}{p} \rfloor$ and $\binom{p+q}{p}$ directly modulo $p = q^x k$ and not q^x .

Remark 2.4. In [4], N. Kayal mentioned that **Theorem 2.1** can be considered to be a very naive and crude primality test.

Remark 2.5. Instead of looking modulo p , we can look at higher powers of p , or we can look at the n -th Fibonacci prime and so on. However, initial investigations by the authors suggest that finding a congruence relation in those cases becomes more difficult.

3. OTHER INTERESTING RESULTS

It is worthwhile to mention without proof a few interesting results that may be deduced from **Lucas' Theorem** or **Theorem 2.1**.

Theorem 3.1. ([5], **Romeo Meštrović, 2009**) If $d, q > 1$ are integers such that,

$$\binom{nd}{md} \equiv \binom{n}{m} \pmod{p}$$

for every pair of integers $n \geq m \geq 0$, then d and q are powers of the same prime p .

Remark 3.2. It follows from the Lucas' Theorem, the congruence relation $\binom{np}{mp} \equiv \binom{n}{m} \pmod{p}$.

Remark 3.3. We do not yet know which are the possible powers of p^k and p^l such that

$$\binom{np^k}{mp^k} \equiv \binom{n}{m} \pmod{p^l}$$

for all integers $n \geq m \geq 1$.

Theorem 3.4. ([8], **David F. Bailey, 1990**) Let n and r be non-negative integers and $p \geq 5$ be a prime. Then,

$$\binom{np}{rp} \equiv \binom{n}{r} \pmod{p^3},$$

where we set $\binom{n}{r} = 0$, if $n < r$.

Theorem 3.5. ([8], **David F. Bailey, 1990**) *Let N, R, n and r be non-negative integers and $p \geq 5$ be a prime. Suppose $n, r < p$, then*

$$\binom{Np^3 + n}{Rp^3 + r} \equiv \binom{N}{R} \binom{n}{r} \pmod{p^3}.$$

Theorem 3.6. ([1], **Tom M. Apostol**) *If p is a prime, α is a positive integer and $p^\alpha \mid \lfloor \frac{n}{p} \rfloor$ then, $p^\alpha \mid \binom{n}{p}$.*

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